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Quasiprobability distribution functions for periodic phase spaces: I. Theoretical aspects

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Abstract

An approach featuring s-parametrized quasiprobability distribution functions is developed for situations where a circular topology is observed. For such an approach, a suitable set of angle–angular momentum coherent states must be constructed in an appropriate fashion.

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1. Introduction

The importance of phase-space quasiprobability distribution functions in the description of different physical systems can hardly be overestimated. Apart from its own theoretical interest [1], they play a key role in quantum optics [2], give an appropriate approach to decoherence [3], insights on semiclassical methods and alternative approaches for dynamics of quantum systems [4].

Quasiprobability distribution functions are defined on quantum phase spaces. A quantum phase-space formalism is generally based on a mapping scheme which enables one to relate operators and functions defined on such phase space [5, 6]. In this kind of approach, the quasiprobability distributions are the functions associated with the density operator. Meanwhile, the plethora of different physical problems might call for different topologies of the phase space itself, on which those functions are defined. However, almost all theoretical techniques regarding quantum phase spaces are suited for the all important case of Cartesian position and momentum variables, where the quantum phase-space domain, spanned by the eigenvalues of position and linear momentum operators, coincides with the classical phase space, a merry coincidence from which much physical knowledge has been inferred. In other scenarios, the situation is somewhat different: for systems where a circular topology is required, the only kind of phase-space quasiprobability distribution function available in the literature is the Wigner function, as proposed by Mukunda [7] and subsequently developed and studied in [8, 9]. It is worth mentioning that the coherent states are scarcely studied in this scenario (in comparison with the Cartesian case) and, as far as the authors' knowledge is concerned, the following are the main references on the subject [10–16].

Returning to the Cartesian realm, the Cahill–Glauber (CG) approach provided a unified and meaningful view of different quasiprobability distribution functions [17]. In addition, this approach is related to particular orderings of operators in a bosonic expansion, paving the way for a better understanding of these functions in the context of quantum optics. However, it is unable to deal with other topologies. In fact, there are no attempts in the literature to deal with the angle–angular momentum phase space (in the sense discussed above) and, consequently, to properly define the appropriate quasiprobability distribution functions.

The aim of this paper is to fill this breach by constructing suitable coherent states in a circle topology which are physically meaningful and closely related with those previously introduced in [10, 13–16]. Basically, the present approach consists in establishing a specific set of algebraic properties that leads us to formally characterize the angle–angular momentum coherent states. The mapping kernel and the associated quasiprobability distribution functions are then properly defined, taking advantage of this algebraic approach and embodying the desired properties of the CG formalism.

This paper is organized as follows. In section 2, we establish the main properties of the angle–angular momentum coherent states which allow us to characterize a quantum phase space with nontrivial topology. In section 3, we define a generalized probability distribution function through a mapping kernel labelled by elements of this phase space where, in particular, the Husimi, Wigner and Glauber–Sudarshan functions are promptly obtained. Moreover, we also derive a hierarchical order among them that consists of a smoothing process described by a well-defined function for the angle–angular momentum variables. Finally, section 4 contains our conclusions and, finally, the appendix presents some basic results on the quantum mechanics of the angle–angular momentum pair, supporting the results of section 2.

2. Algebraic properties of the angle–angular momentum coherent states

To construct the angle–angular momentum coherent states, we first introduce a normalized reference state (or vacuum state) through a continuous superposition of angle eigenstates (which are discussed in the appendix, along with other pertinent details), namely

$$|0,0\rangle_{\mathrm{B}} \equiv \int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathfrak{F}_{\mathrm{B}}(\theta)|\theta\rangle, \tag{1}$$

with complex coefficients

$$\mathfrak{F}_{\mathrm{B}}(\theta) \equiv \langle\theta|0,0\rangle_{\mathrm{B}} = \frac{1}{\sqrt{2\pi}} \frac{\vartheta_3\left(\frac{\theta}{2}\middle|\mathrm{i}\mathfrak{a}\right)}{\sqrt{\vartheta_3(0|2\mathrm{i}\mathfrak{a})}} \qquad \text{(boson case)}$$

evaluated in terms of the Jacobi theta functions [18] for $\mathfrak{a} = (2\pi)^{-1}$. The Jacobi ϑ_3-function itself reads as

$$\vartheta_3(z|\tau) = \sum_{l=-\infty}^{\infty} \exp[\mathrm{i}\pi\tau l^2]\exp[2\mathrm{i}lz]. \tag{2}$$

The Jacobi ϑ_3-function can be obtained by the Poisson sum applied to the Gaussian function [19]. Note that the second argument of the ϑ_3-function controls its width, and with the value chosen here the normalized vacuum state coincides with that proposed in [10, 13]. Now, when m assumes half-integer values, the complex coefficients $\mathfrak{F}_{\mathrm{F}}(\theta) \equiv \langle\theta|0,0\rangle_{\mathrm{F}}$ must be written as follows:

$$\mathfrak{F}_{\mathrm{F}}(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\vartheta_2\left(\frac{\theta}{2}\middle|\mathrm{i}\mathfrak{a}\right)}{\sqrt{\vartheta_2(0|2\mathrm{i}\mathfrak{a})}} \qquad \text{(fermion case),}$$

and the ϑ_2-function by its turn is

$$\vartheta_2(z|\tau) = \sum_{l=-\infty}^{\infty} \exp\left[\mathrm{i}\pi\tau\left(l+\frac{1}{2}\right)^2\right]\exp\left[2\mathrm{i}\left(l+\frac{1}{2}\right)z\right]. \tag{3}$$

For convenience, we will particularize our results for $m \in \mathbb{Z}$ (boson case) throughout this work.

The next step is to adopt Klauder's prescription for coherent states [20] through the use of unitary displacement operators, i.e.,

$$|m, \theta\rangle \equiv \mathbf{D}(m, \theta)|0, 0\rangle, \tag{4}$$

where

$$\mathbf{D}(m, \theta) = \exp\left(-\frac{\mathrm{i}}{2}m\theta\right)\exp(\mathrm{i}m\Theta)\exp(-\mathrm{i}\theta\mathbf{J}). \tag{5}$$

It is always pertinent to remember that, as it is constructed, the angular unitary displacement operator obeys the periodicity required for angular eigenstates. It is also worth mentioning some basic properties of the $\mathbf{D}(m, \theta)$ displacement operators, which follow from appropriate use of equation (A.4):

(i) $\mathbf{D}^{\dagger}(m, \theta) = \mathbf{D}(-m, -\theta)$,

(ii) $\mathrm{Tr}[\mathbf{D}^{\dagger}(m', \theta')\mathbf{D}(m, \theta)] = \delta_{m',m}\delta(\theta' - \theta)$,

the second property being obtained by means of the multiplication law

$$\mathbf{D}(m', \theta')\mathbf{D}(m, \theta) = \exp\left[\frac{\mathrm{i}}{2}\left(m'\theta - m\theta'\right)\right]\mathbf{D}(m + m', \theta + \theta').$$

In addition, the set of coherent states $\{|m, \theta\rangle\}$ satisfies two important properties associated with the completeness relation and the scalar product of two angle–angular momentum coherent states, namely

$$\text{(iii)} \quad \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} |m, \theta\rangle\langle m, \theta| = \mathbf{1}$$

and

$$\text{(iv)} \quad \langle m', \theta'|m, \theta\rangle = \exp\left\{-\frac{1}{2}(m - m')^2 + \frac{\mathrm{i}}{2}[(m\theta' - m'\theta) + (m - m')(\theta - \theta')]\right\} \times \frac{\vartheta_3\left(\frac{1}{2}(\theta - \theta') + \frac{\mathrm{i}}{2}(m - m')\middle|2\mathrm{i}\mathfrak{a}\right)}{\vartheta_3\left(0|2\mathrm{i}\mathfrak{a}\right)}.$$

The sum (instead of an integral) over all integer values of angular momentum in property (iii) asserts that the quantum phase space for angular coordinates is not equivalent to the classical one [7]. In order to prove this equality, one needs only to decompose the coherent states in either the angle or the angular momentum basis and properly identify the realization of Dirac or Kroenecker deltas, observing the periodicity of the angle variable or the infinite range of the angular momentum. Moreover, property (iv), which can be directly (but tediously) obtained, presents a perfect analogy with the Cartesian case, where the ϑ_3-function plays in this context the role that is reserved to the Gaussian function in the linear case. In fact, the angle–angular momentum coherent states constructed here present a complete analogy (apart, at least, from a phase factor) with those previously introduced in [10, 13–16].

2.1. Uncertainty relations

To avoid some mathematical inconsistencies in the uncertainty relations following the commutation relation between the $\mathbf{J}$ and Θ operators, Carruthers and Nieto [21], following Louisell [22], introduced $\sin(\Theta)$ and $\cos(\Theta)$ which inherently embodies the periodicity property. For parity reasons, that appropriate to be concerned here is the non-symmetrical relation

$$U \equiv \langle\Delta\mathbf{J}\rangle^2\langle\Delta \sin(\boldsymbol{\Theta})\rangle^2 \geqslant \tfrac{1}{4}\langle\cos(\boldsymbol{\Theta})\rangle^2, \tag{6}$$

where the variances are explicitly evaluated through the relation $\langle\Delta\mathbf{O}\rangle^2 \equiv \langle\mathbf{O}^2\rangle - \langle\mathbf{O}\rangle^2$. Now, let us consider the coherent states (4) and their algebraic properties in this context. For instance, it is straightforward to show that (6) does not depend on the angular-momentum label, namely $U \equiv U(\theta)$ (the symmetrical inequality given by Carruthers and Nieto is seen to be independent of both variables). Besides, through the auxiliary relation

$$\delta U(\theta) \equiv \frac{\langle\Delta\mathbf{J}\rangle^2\langle\Delta \sin(\Theta)\rangle^2 - (1/4)\langle\cos(\Theta)\rangle^2}{\langle\Delta\mathbf{J}\rangle^2\langle\Delta \sin(\Theta)\rangle^2},$$

it is possible to verify that for different values of $\mathfrak{a}$ (width of the ϑ_3-function associated with the normalized reference state) and θ the angle–angular momentum coherent states are minimum uncertainty states. Figure 1 shows the plots of $\delta U(\theta)$ versus θ for (*a*) $\mathfrak{a} = 1/20\pi$, (*b*) $\mathfrak{a} = 1/2\pi$ (value adopted in this work) and (*c*) $\mathfrak{a} = 10/2\pi$. Note that $\delta U(\theta)$ reaches its maximum value at the points $\theta = \pm\pi/2$ in all pictures, while its minimum value occurs at the points $\theta = 0, \pm\pi$. Since minimum uncertainty states are characterized by the mathematical condition $\delta U(\theta) = 0$, we can perceive that (*b*) shows in this case a small deviation around 4% for the points located in $\theta = 0, \pm\pi$. Consequently, the right choice of parameters $\mathfrak{a}$ and θ leaves relation (6) arbitrarily close to the equality (for similar results, see [10]).

3. Mapping kernel

Following the mathematical procedure established in [24] (where the exposition, although in another context, is more detailed and instructive), given a set of coherent states it is always possible to define the mapping kernel

$$\mathbf{T}^{(s)}(m,\theta) \equiv \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi} \mathrm{d}\alpha \exp[-\mathrm{i}l(\theta-\Theta)]\exp[\mathrm{i}\alpha(m-\mathbf{J})]\exp\left(-\frac{\mathrm{i}}{2}l\alpha\right)[\mathcal{K}(l,\alpha)]^{-s}, \tag{7}$$

where $\mathcal{K}(l,\alpha) \equiv \langle 0,0|l,\alpha\rangle$ denotes a particular overlap of coherent states explicitly calculated in (iv) and s represents a complex parameter satisfying $|s| \leqslant 1$. It is easy to show that the properties

(v) $\mathrm{Tr}[\mathbf{T}^{(s)}(m,\theta)] = 1$,

(vi) $\mathrm{Tr}[\mathbf{T}^{(-s)}(m',\theta')\mathbf{T}^{(s)}(m,\theta)] = 2\pi\delta_{m',m}\delta(\theta'-\theta)$

are promptly verified where, in particular, the last equality has been attained with the help of the auxiliary relation

$$\mathrm{Tr}[\mathbf{T}^{(s')}(m',\theta')\mathbf{T}^{(s)}(m,\theta)] = \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}\mathrm{d}\alpha \exp\{\mathrm{i}[l(\theta'-\theta)-\alpha(m'-m)]\}[\mathcal{K}(l,\alpha)]^{-(s'+s)}. \tag{8}$$

Indeed, property (vi) guarantees that the decomposition of any bounded operator $\mathbf{O}$ in this basis assumes the form

$$\mathbf{O} = \sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\mathcal{O}^{(-s)}(m,\theta)\mathbf{T}^{(s)}(m,\theta), \tag{9}$$

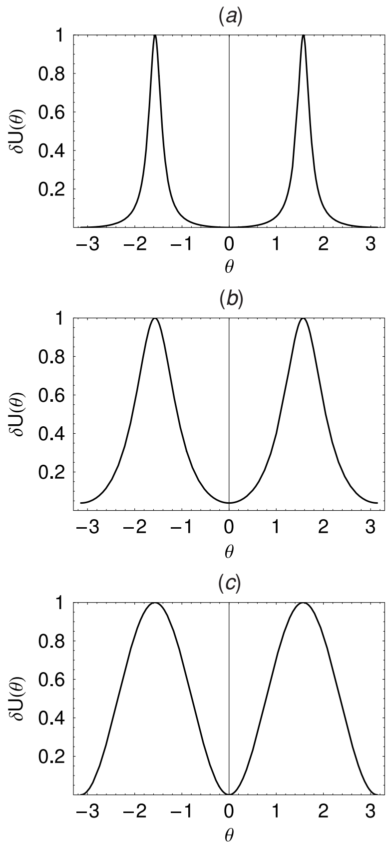

Figure 1. Plots of $\delta U(\theta)$ versus $\theta \in [-\pi, \pi)$ for (*a*) $\mathfrak{a} = 1/20\pi$, (*b*) $\mathfrak{a} = 1/2\pi$ and (*c*) $\mathfrak{a} = 10/2\pi$. These pictures show that $0 \leqslant \delta U(\theta) \leqslant 1$, and for $\delta U(\theta) = 0$ the coherent states $\{|m, \theta\rangle\}$ can be considered as minimum uncertainty states. It is worth mentioning that the main differences between intelligent and minimum uncertainty states have been discussed in [23] for the experimental context.

where the coefficients

$$\mathcal{O}^{(-s)}(m, \theta) \equiv \mathrm{Tr}[\mathbf{T}^{(-s)}(m, \theta)\mathbf{O}]$$

correspond to a one-to-one mapping between operators and functions belonging to a phase space characterized by the angle–angular momentum variables. In addition, the mean value

$$\langle \mathbf{O} \rangle \equiv \mathrm{Tr}(\mathbf{O}\boldsymbol{\rho}) = \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \mathcal{O}^{(-s)}(m, \theta) \mathcal{F}^{(s)}(m, \theta) \tag{10}$$

can also be obtained from this decomposition, the parametrized function $\mathcal{F}^{(s)}(m, \theta)$ being defined as the expectation value of the mapping kernel (7), i.e.,

$$\mathcal{F}^{(s)}(m, \theta) \equiv \mathrm{Tr}[\mathbf{T}^{(s)}(m, \theta)\boldsymbol{\rho}] \tag{11}$$

with $\boldsymbol{\rho}$ representing the density operator which describes an arbitrary physical system. As it will be seen, for $s = -1, 0, +1$ the parametrized function is directly related to the Husimi, Wigner and Glauber–Sudarshan functions, respectively.

3.1. Quasiprobability distribution functions

Now, let us consider the expansion for the projector of angle–angular momentum coherent states in the basis $\{\mathbf{D}(l,\alpha)\}$ as follows:

$$|m,\theta\rangle\langle m,\theta| = \sum_{l\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\alpha}{2\pi} \mathbf{D}(l,\alpha)\,\mathrm{Tr}[\mathbf{D}^{\dagger}(l,\alpha)|m,\theta\rangle\langle m,\theta|], \tag{12}$$

where the Weyl algebra, obeyed by the displacement operators, ensures that

$$\mathrm{Tr}[\mathbf{D}^{\dagger}(l,\alpha)|m,\theta\rangle\langle m,\theta|] = \exp[-\mathrm{i}(l\theta - m\alpha)]\mathcal{K}(l,\alpha). \tag{13}$$

Now, using the above result for the trace in equation (12), one immediately realizes that the obtained expression is exactly the same as that of equation (7) for the particular case $s=-1$, which allows for the conclusion that

$$|m,\theta\rangle\langle m,\theta| = \mathbf{T}^{(-1)}(m,\theta), \tag{14}$$

which perhaps is the central result of this paper. A lot of consequences follow from this fact. First, the Husimi function can be defined within this context by means of a trace operation,

$$\mathcal{H}(m,\theta) \equiv \mathcal{F}^{(-1)}(m,\theta) = \mathrm{Tr}[|m,\theta\rangle\langle m,\theta|\boldsymbol{\rho}] = \langle m,\theta|\boldsymbol{\rho}|m,\theta\rangle. \tag{15}$$

On the other hand, if one considers $s=-1$ and $\mathbf{O}=\rho$ in equation (9), we obtain the diagonal representation

$$\boldsymbol{\rho} = \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \mathcal{P}(m,\theta)|m,\theta\rangle\langle m,\theta|, \tag{16}$$

where $\mathcal{P}(m,\theta) \equiv \mathcal{F}^{(1)}(m,\theta)$ plays the role of the Glauber–Sudarshan function.

In particular, for $s=0$ the mapping kernel reduces to the form already studied in [7–9]. Hence, $\mathcal{F}^{(0)}(m,\theta) \equiv \mathcal{W}(m,\theta)$ is nothing else but the Wigner function associated with the angle–angular momentum representation.

3.2. Hierarchical structure

Next, we will obtain a hierarchical structure relating the Glauber–Sudarshan, Wigner and Husimi quasiprobability distribution functions. For this purpose, let us initially decompose the element $\mathbf{T}^{(s)}(m,\theta)$ with the help of equation (9),

$$\mathbf{T}^{(s)}(m,\theta) = \sum_{m'\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta'}{2\pi}\,\mathrm{Tr}[\mathbf{T}^{(-s')}(m',\theta')\mathbf{T}^{(s)}(m,\theta)]\mathbf{T}^{(s')}(m',\theta'). \tag{17}$$

The expression for $\mathrm{Tr}[\mathbf{T}^{(-s')}(m',\theta')\mathbf{T}^{(s)}(m,\theta)] \equiv \mathfrak{Z}^{(s'-s)}(m'-m,\theta'-\theta)$ is explicitly shown in equation (8). An immediate consequence of this result is the link between different parametrized functions

$$\mathcal{F}^{(s)}(m,\theta) = \sum_{m'\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta'}{2\pi} \mathfrak{Z}^{(s'-s)}(m'-m,\theta'-\theta)\mathcal{F}^{(s')}(m',\theta'), \tag{18}$$

where the term $\mathfrak{Z}^{(s'-s)}(m'-m,\theta'-\theta)$ plays an important role in this process. Note that the double Fourier transform on the right-hand side of equation (8) is well defined only for $\mathrm{Re}(s'-s) \geqslant 0$; otherwise, the infinite summation for the angular momentum variable gives a divergent result. This implies in a hierarchical structure analogous to that observed for the Cartesian case (we remark that this is not observed in the extended CG formalism for finite-dimensional spaces [24]).

Two important results can be promptly reached through equation (18) for specific values of the complex parameters s and s', i.e.,

$$\mathcal{W}(m,\theta) = \sum_{m'\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta'}{2\pi} \mathfrak{Z}^{(1)}(m'-m,\theta'-\theta)\mathcal{P}(m',\theta'), \tag{19}$$

$$\mathcal{H}(m,\theta) = \sum_{m'\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta'}{2\pi} \mathfrak{Z}^{(1)}(m'-m,\theta'-\theta)\mathcal{W}(m',\theta'). \tag{20}$$

Therefore, equations (19) and (20) exhibit a sequential smoothing which characterizes a hierarchical process among the quasiprobability distribution functions in the angle–angular momentum phase space, $\mathcal{P}(m,\theta) \to \mathcal{W}(m,\theta) \to \mathcal{H}(m,\theta)$. Here,

$$\mathfrak{Z}^{(1)}(m'-m,\theta'-\theta) \equiv \mathrm{Tr}[\mathbf{T}^{(-1)}(m',\theta')\mathbf{T}^{(0)}(m,\theta)] = \langle m',\theta'|\mathbf{T}^{(0)}(m,\theta)|m',\theta'\rangle$$

can be interpreted as a Wigner function evaluated for the angle–angular momentum coherent states labelled by m' and θ'. It is worth mentioning that

$$\mathcal{H}(m,\theta) = \sum_{m'\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta'}{2\pi} |\langle m',\theta'|m,\theta\rangle|^2 \mathcal{P}(m',\theta') \tag{21}$$

establishes an additional relation which allows us to connect both the Husimi and Glauber–Sudarshan functions without the intermediate process given by the Wigner function, with $|\langle m',\theta'|m,\theta\rangle|^2 = |\mathcal{K}(m'-m,\theta'-\theta)|^2$ being the overlap probability for coherent states.

4. Conclusions

One of the interesting features of the original CG approach is that it gives a clear-cut answer to the problem of ordered expansions in boson amplitude operators. As the own creation and annihilation operators in the present context are not a closed matter, we leave this question to be discussed elsewhere, as our results might provide a useful approach to this problem.

In a more pragmatical sense, the above discussed quasiprobability distribution functions can be seen to be tailored for use in treating the rotational degree of freedom in deformed physical systems and discussing their semiclassical limits. In particular, some previous attempts of introducing rotational coherent states have been put forth in the past whose aim was to treat the dynamics of two-dimensional deformed systems in molecular physics [25]. In this connection, the use of the distributions proposed here in such studies of rigid deformed systems dynamics, in the context of von Neumann–Liouville formalism, seems to be a promising perspective.

Our results also seem to be quite suitable to deal with the problem of quantum rings, where a single electron can be trapped in a region whose topology is exactly that regarded here [26]. It is reasonable to expect that the quasiprobability functions might provide convenient tools for obtaining physical information from such systems.

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Appendix. Quantum mechanics of the angle–angular momentum pair

Let us initially consider a vector space spanned by an infinite set of states $\{|m\rangle\}_{m\in\mathbb{Z}}$ obeying

$$\mathbf{J}|m\rangle = m|m\rangle, \tag{A.1}$$

where we take full advantage of the abstract Dirac notation. Here, for briefness, we deal only with the boson case. This set of states is thus orthogonal, $\langle m'|m\rangle = \delta_{m',m}$, and complete by assumption,

$$\sum_{m=-\infty}^{\infty} |m\rangle\langle m| = \mathbf{1}.$$

We then introduce a new family of states, constructed making use of the Fourier coefficients

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \exp(\mathrm{i}m\theta)|m\rangle,$$

where the label θ is a real number and, by construction, $|\theta\rangle = |\theta + 2\pi\rangle$ (therefore, one can always work within the interval $-\pi \leqslant \theta < \pi$). It is possible to verify that these states are orthogonal and complete,

$$\langle\theta'|\theta\rangle = \delta^{[2\pi]}(\theta' - \theta), \qquad \int_{-\pi}^{\pi} \mathrm{d}\theta |\theta\rangle\langle\theta| = \mathbf{1},$$

where the superscript 2π in the Dirac delta denotes that it is different from zero whenever $\theta' - \theta = 0 \pmod{2\pi}$ (or, in a somewhat clumsier notation, $\delta^{[2\pi]}(\theta' - \theta) = \sum_{k=-\infty}^{\infty} \delta(\theta' - \theta + 2\pi k)$).

One may now define an angle operator by means of a spectral decomposition

$$\mathbf{\Theta} = \int_{-\pi}^{\pi} \mathrm{d}\theta\, \theta |\theta\rangle\langle\theta|$$

and observe that the exponentials of these operators act as

$$\exp(-\mathrm{i}\theta\mathbf{J})|\theta'\rangle = |\theta' + \theta\rangle, \tag{A.2}$$

$$\exp(\mathrm{i}m\mathbf{\Theta})|m'\rangle = |m' + m\rangle. \tag{A.3}$$

From the above results follows that Weyl algebra is observed,

$$\exp(\mathrm{i}\theta\mathbf{J}) \exp(\mathrm{i}m\mathbf{\Theta}) = \exp(\mathrm{i}m\theta) \exp(\mathrm{i}m\mathbf{\Theta}) \exp(\mathrm{i}\theta\mathbf{J}). \tag{A.4}$$

Therefore, these operators are the displacement generators in each other's set of eigenstates—which are connected through a Fourier transform, and it is exactly in this sense that one can say that they are canonically conjugated.

To see a typical realization of the action of the angular momentum operator, if one considers an arbitrary state $|\psi\rangle$,

$$|\psi\rangle = \int_{-\pi}^{\pi} \mathrm{d}\theta\, \psi(\theta)|\theta\rangle,$$

where $\psi(\theta)$ is an arbitrary periodic function continuous in the interval $[-\pi, \pi)$, the state $\mathbf{J}|\psi\rangle$ is seen to be represented by

$$\mathbf{J}|\psi\rangle = \int_{-\pi}^{\pi} \mathrm{d}\theta\, \psi(\theta)\mathbf{J}|\theta\rangle$$

and convenient use of the above resolutions of unity leads to

$$\mathbf{J}|\psi\rangle = \int_{-\pi}^{\pi} \mathrm{d}\theta' \left[\int_{-\pi}^{\pi} \mathrm{d}\theta\, \psi(\theta) \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} m \exp(\mathrm{i}m(\theta - \theta')) \right] |\theta'\rangle.$$

Calling I the term inside the square brackets, one sees that it is in fact

$$I = -\mathrm{i} \int_{-\pi}^{\pi} \mathrm{d}\theta\, \psi(\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp(\mathrm{i}m(\theta - \theta')) = -\mathrm{i} \int_{-\pi}^{\pi} \mathrm{d}\theta\, \psi(\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} \delta^{[2\pi]}(\theta - \theta'),$$

as $\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}m(\theta - \theta')]$ is a representation of the modulo 2π Dirac delta. Integration by parts (and the periodicity of $\psi(\theta)$) then leads to

$$I = \mathrm{i} \frac{\mathrm{d}\psi(\theta')}{\mathrm{d}\theta'},$$

which means that

$$\mathbf{J}|\psi\rangle = \int_{-\pi}^{\pi} \mathrm{d}\theta' \left[\mathrm{i} \frac{\mathrm{d}\psi(\theta')}{\mathrm{d}\theta'} \right] |\theta'\rangle,$$

and thus $\mathbf{J}$ is seen as a derivative operator in the angle representation, once one is dealing with states $|\psi\rangle$ constructed out of periodic functions. One can then, in principle, look for the commutator $[\mathbf{J}, \boldsymbol{\Theta}]$, although it is no trivial matter to obtain an uncertainty relation from it, as was discussed in [21]. We remark that, however, the well-defined Weyl commutation, obeyed by the displacement operators, is all the algebra necessary for the purpose of this paper.

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